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Coupled Librational Motions of a Gyrostat Satellite

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This paper deals with coupled librational motions of a rigid gyrostat satellite in an elliptic orbit. The analysis is based on the assumptions that the orbital eccentricity is small and that the satellite is an aspherical rigid body with one or more rigidly attached gyro-rotors. The differential equations of the Euler angles are derived for the perturbed motion from gravitational equilibria in a circular orbit. Using the linearized system of equations as the basis, the equation of natural frequencies, and harmonic response are formulated. Periodic solutions of eccentricity forced motion are obtained by using a two-term Fourier series for the pitch motion and power series in eccentricity for the roll and yaw motions. A rigid gyrostat satellite having flexible appendages is also treated.

I. Introduction

PRECISE attitude control of an Earth-pointed orbiting satellite is very important to many space projects. The use of gyro-rotors as means of attitude control and stability analysis of the attitude motions of a satellite in a circular orbit have attracted numerous researchers in the last decade. The research works may be roughly divided into two categories according to the physical system of a satellite, namely, gyrostat satellites[1–8] and spinning satellite with flexible appendages.[9–11]

Effect of a rotor on the attitude stability of a gyrostat satellite was analyzed by Kane and Mingori[1] using Floquet theory. Breakwell and Pringle[2] applied the Hamilton-Jacobi theory to treat nonlinear resonance of a gyrostat satellite. The existence of four cases of gravitational equilibria for a rigid gyrostat satellite in a circular orbit was established by Roberson and Hooker[3] and the general solutions of these equilibria were obtained by Longman and Roberson.[4] The stability analysis of all possible equilibria were presented first by Rumyantsev[5] by using the principle of stationary points of a transformed potential energy and then by Longman[6] by Liapunov method. The infinitesimally stable and Liapunov stable regions for case 1 equilibria, as defined by Ref. 3, were presented in three dimensional plots of vehicle shape parameters by Crespo da Silva.[7] In Ref. 8, he also investigated nonlinear resonant attitude motions about case 1 equilibria both analytically and numerically.

Space vehicle control engineers are concerned with the existence of high frequency components of an orbiting vehicle in ground laboratory simulation tests. Great efforts, therefore, have been devoted to structural analysis of a flexible primary body connected with many elastic sub-bodies. This approach often requires lengthy computations and results in a large number of variables.

Stability analysis of a nonsymmetric gyrostat satellite for the cases 2 and 3 equilibria involves five parameters, instead of three for case 1. Determination of stable regions and nonlinear resonance by the method such as that used in Refs. 7 and 8 becomes a cumbersome task. The first part of this paper presents a study of a linearized attitude motions for cases 2 and 3 equilibria of a gyrostat satellite in a circular orbit. The second part concerns with solutions for the eccentricity forced attitude motions if the satellite orbit has a small eccentricity. The last part gives a simplified approach for dealing with dynamic coupling of a flexible appendage.

In this analysis we assume that: 1) a gyrostat satellite consists of an aspherical rigid primary body and one or more dynamically balanced rotors; 2) the axes of the rotors are rigidly attached to the primary body and the rotors have constant speeds relative to the primary body; and 3) for a satellite in an elliptical orbit, the magnitude of eccentricity is of the same order as the perturbed variables.

II. Analysis

A. Derivation of Equations of Motion

Let us consider that a gyrostat satellite consists of a rigid primary body and one or more dynamically balanced gyro-rotors

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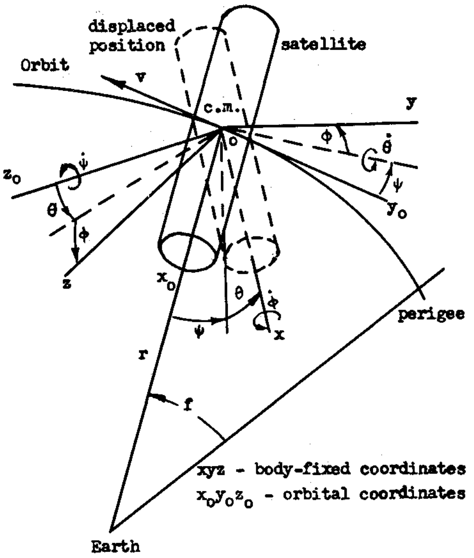

Fig. 1 The Euler angles and the coordinates.

which are rigidly attached to the primary body. The Euler equations of rotational motion about its center of mass under the influence of gravity-gradient moments of the Earth and disturbing torques are

$$\begin{aligned}\dot{\omega}_x &= -K_1\omega_y\omega_z+3(\mu/r^3)K_1\beta\gamma+(Y_1\omega_z-Z_1\omega_y)n_o+T_x\\ \dot{\omega}_y &= -K_2\omega_x\omega_z+3(\mu/r^3)K_2\gamma\alpha+(Z_2\omega_x-X_2\omega_z)n_o+T_y \qquad (1)\\ \dot{\omega}_z &= -K_3\omega_x\omega_y+3(\mu/r^3)K_3\alpha\beta+(X_3\omega_y-Y_3\omega_x)n_o+T_z\end{aligned}$$

where dot denotes time derivative and subscripts, x, y, and z, designate components along the principal axes. Other symbols are defined in the following:

1) The coordinate systems and the Euler angles

The orbital coordinate $x_o y_o z_o - x_o$ is directed toward the center of the Earth, y_o is opposite the velocity, and z_o is normal to the orbital plane.

The body-fixed coordinates $xyz - xyz$ are principal axes of a satellite with origin at the center of mass.

The Euler angles ψ, θ, and Φ—the angles are the successive angular rotations of the body-fixed coordinates relative to the orbital coordinates. As shown in Fig. 1, first, a rotation ψ about the axis aligned with z_o, then rotation θ about the new position of the axis aligned with y, and Φ about the final position of the axis aligned with x.

The direction cosines of the principal axes xyz relative to the orbital radius vector are

$$\begin{aligned}\alpha &= -\cos\psi\cos\theta\\ \beta &= \sin\psi\cos\Phi-\cos\psi\sin\theta\sin\Phi \qquad (2)\\ \gamma &= -\sin\psi\sin\Phi-\cos\psi\sin\theta\cos\Phi\end{aligned}$$

2) The inertia parameters

$$K_1=(I_z-I_y)/I_x, \quad K_2=(I_x-I_z)/I_y$$
$$K_3=(I_y-I_x)/I_z = -(K_1+K_2)/(1+K_1K_2)$$

where I_x, I_y, and I_z are the moments of inertia about the principal axes x, y, and z, respectively. Note that although K_3 is not an independent parameter, it will be kept for simplicity.

3) The angular velocity components along the principal axes

$$\begin{aligned}\omega_x &= \dot{\Phi}-(\dot{\psi}+\dot{f})\sin\theta\\ \omega_y &= (\dot{\psi}+\dot{f})\cos\theta\sin\Phi+\dot{\theta}\cos\Phi \qquad (3)\\ \omega_z &= (\dot{\psi}+\dot{f})\cos\theta\cos\Phi-\dot{\theta}\sin\Phi\end{aligned}$$

4) The disturbing torques and angular momenta of rotors

$$T_x = x\text{-component of the disturbing torque}/I_x, \text{ etc.}$$
$$X_2 = h_x/I_y n_o, \quad Z_1 = h_z/I_x n_o, \text{ etc.}$$

where h is the angular momentum of the rotor relative to the primary body.

5) The orbital constants and elements used in Eqs. (1–3) and later equations

For these, μ = gravity constant of the earth; r = orbital radius; f = true anomaly; M = mean anomaly; n_o = mean orbital angular velocity $= (\mu/a^3)^{1/2}$; a = semimajor axis; and e = eccentricity.

For the convenience of using series solution as well as making the differential equations dimensionless, we use the mean anomaly, M, as the independent variable and let "prime" denote the derivative with respect to M. Then

$$d/dt = n_o\, d/dM, \qquad d^2/dt^2 = n_o^2 d^2/dM^2$$

Substituting Eqs. (2) and (3) into Eq. (1), we obtain, after some simple manipulations

$$\begin{aligned}&\psi''+f''-\theta'\Phi'\sec\theta-\theta'(\psi'+f')\tan\theta = F\\ &\Phi''-\theta'(\psi'+f')\sec\theta-\theta'\Phi'\tan\theta = F\sin\theta+\\ &\quad \tfrac{1}{2}K_1\{[(\psi'+f')^2\cos^2\theta-(\theta')^2]\sin 2\Phi+ \qquad (4)\\ &\quad 2(\psi'+f')\theta'\cos\theta\cos 2\Phi-3\bar{a}[(\cos^2\psi\sin^2\theta-\sin^2\psi)\times\\ &\quad \sin 2\Phi-\sin\theta\sin\psi\cos 2\Phi]\}+(Y_1\cos\Phi-Z_1\sin\Phi)\times\\ &\quad (\psi'+f')\cos\theta-(Y_1\sin\Phi+Z_1\cos\Phi)\theta'+u_x\\ &\theta''+(\psi'+f')\Phi'\cos\theta = (K_3\sin^2\Phi-K_2\cos^2\Phi)\times\\ &\quad [(\psi'+f')p-3\bar{a}\cos^2\psi\sin\theta]\cos\theta+\\ &\quad \tfrac{1}{2}(K_3+K_2)[p\theta'+\tfrac{3}{2}\bar{a}\sin 2\psi\cos\theta]\sin 2\Phi+\\ &\quad (Z_2\cos\Phi+Y_3\sin\Phi)p+(X_2-X_3)\theta'\sin\Phi\cos\Phi-\\ &\quad (X_3\sin^2\Phi+X_2\cos^2\Phi)(\psi'+f')\cos\theta+u_y\cos\Phi-u_z\sin\Phi\end{aligned}$$

where

$$\bar{a} = (a/r)^3$$
$$p = \Phi'-(\psi'+f')\sin\theta$$
$$\begin{aligned}F = &-(K_3\cos^2\Phi-K_2\sin^2\Phi)[p\theta'\sec\theta+\tfrac{3}{2}\bar{a}\sin 2\psi]-\\ &\tfrac{1}{2}(K_2+K_3)[(\psi'+f')p-3\bar{a}\cos^2\psi\sin\theta]\sin 2\Phi+\\ &(Z_2\sin\Phi-Y_3\cos\Phi)\,p\sec\theta-\tfrac{1}{2}(X_2-X_3)(\psi'+f')\sin 2\Phi-\\ &(X_2\sin^2\Phi+X_3\cos^2\Phi)\theta'\sec\theta+\\ &(u_y\sin\Phi+u_z\cos\Phi)\sec\theta\end{aligned}$$
$$u_x = T_x/n_o^2, \text{ etc.}$$

Series expansions of the orbital elements, f', f'', and $(a/r)^3$, in terms of the mean anomaly, M, and eccentricity, e, are readily obtained

$$\begin{aligned}f' &= 1-e^2(a/r)^2 = 1+2e\cos M+\tfrac{5}{2}e^2\cos 2M+\cdots\\ -f'' &= 2e(a/r)^3\sin f = 2e\sin M+5e^2\sin 2M+\cdots \qquad (5)\\ (a/r)^3 &= 1+\tfrac{3}{2}e^2+3e\cos M+\tfrac{9}{2}e^2\cos 2M+\cdots\end{aligned}$$

The equations of the coupled motion are very complex and nonlinear. For practical purposes, we may assume the angular displacements from a satellite's equilibrium states are small and our main concern is the dynamic characteristics of a linearized system of equations. The linearized equations of attitude motions about all the cases of equilibria, as defined by Ref. 3, can be obtained from Eq. (4) by setting appropriate values for the Euler angles and the angular momentum components of the rotors. The case 4 equilibria will not be treated here.

1) Case 1: equilibrium state

This state requires that the relative angular momentum vector of the rotors be aligned with the z-axis, i.e., $X_i = Y_i = 0$, (i = 1, 2, 3,), and that the equilibrium Euler angles be $\psi_o = \Phi_o = \theta_o = 0$. We treat the variables in Eq. (4) as small perturbations and consider the eccentricity and the disturbing torques the same order of magnitude as the perturbed variables. Linearization of Eq. (4) yields

$$\psi'' + 3K_3\psi = 2e\sin M + u_z$$
$$\Phi'' + b\theta' + a\Phi = u_x \quad (6)$$
$$\theta'' + c\Phi' + d\theta = u_y$$

where $a = K_1 + Z_1$, $b = K_1 - 1 + Z_1$, $c = 1 + K_2 - Z_2$, and $d = Z_2 - 4K_2$.

The variable ψ is not coupled with Φ and θ and the natural frequencies of the system can be obtained by assuming the solution, $\Phi = A\cos\lambda M$ and $\theta = B\sin\lambda M$, for Eq. (6). This yields two natural frequencies

$$\omega_{1,2} = n_o\lambda_{1,2} = (2)^{1/2}/2\{a+d-bc \pm [(a+d-bc)^2 - 4ad]^{1/2}\}^{1/2} n_o \quad (7)$$

For stable motion, or for $\omega_{1,2}{}^2$ to be real and positive, the conditions are

$$a+d-bc > 0, \qquad (a+d-bc)^2 > 4ad > 0 \quad (8)$$

Note that Eq. (7) agrees with the result given by Ref. 7 and the corresponding notations are: $Z_3 = \beta$, $Z_1 = \beta(1+K_1K_2)/(1+K_2)$, and $Z_2 = \beta(1+K_1K_2)/(1-K_1)$.

An interesting case is that of a satellite having a single, large, high speed rotor, i.e., $Z_1 \gg 1$ and $Z_2 \gg 1$. If the order of magnitude of K_2 and K_1 is unity, the natural frequencies reduce to

$$\omega_1 = n_o, \qquad \omega_2 = \Omega I_r/(I_x I_y)^{1/2} \quad (9)$$

where Ω and I_r denote, respectively, relative angular velocity and moment of inertia of the rotor. It is remarkable that the natural frequency for liberational motion may have the same order of magnitude as the angular velocity of the rotor.

2) Case 2: equilibrium state

For this case, the y-component of the relative angular momentum vector of the gyro-rotor vanishes, i.e., $Y_i = 0$ with $i = 1, 2, 3$, and the Euler angles are $\psi_o = \Phi_o = 0$ and $\theta_o \neq 0$. The linearized system of equations for the perturbed variables obtained from Eq. (4) is

$$\psi'' + a_{11}\psi + a_{12}\Phi - a_{13}\delta\theta' = 2e\sin M + u_z\sec\theta_o$$
$$a_{12}\psi + \Phi'' + a_{22}\Phi - a_{23}\delta\theta' = u_x + u_z\tan\theta_o \quad (10)$$
$$a_{31}\psi' + a_{32}\Phi' + \delta\theta'' + a_{33}\delta\theta = u_y + 2K_2\sin 2\theta_o - Z_2\sin\theta_o - X_2\cos\theta_o$$

where $\delta\theta = \theta - \theta_o$, and the a_{ij} are presented in Appendix A. Since $\delta\theta = 0$ is the equilibrium state, the constant term on the right-hand side of the third equation must vanish

$$2K_2\sin 2\theta_o = Z_2\sin\theta_o + X_2\cos\theta \quad (11)$$

from which the equilibrium angle θ_o can be determined.

3) Case 3: equilibrium state

This state corresponds to $X_i = 0$ with $i = 1$, 2, 3, and $\psi_o = \theta_o = 0$ and $\Phi_o \neq 0$. The linearized equations about the equilibrium state are

$$\psi'' + a_{11}\psi + a_{12}\theta - a_{13}\delta\Phi' = 2e\sin M + u_y\sin\Phi_o + u_z\cos\Phi_o$$
$$a_{21}\psi + \theta'' + a_{22}\theta - a_{23}\delta\Phi' = u_y\cos\Phi_o - u_z\sin\Phi_o \quad (12)$$
$$a_{31}\psi' + a_{32}\theta' + \delta\Phi'' + a_{33}\delta\Phi = u_x - \tfrac{1}{2}K_1\sin 2\Phi_o + Y_1\cos\Phi_o - Z_1\sin\Phi_o$$

where $\delta\Phi = \Phi - \Phi_o$, and the constants a_{ij} are given in Appendix A. Setting the constant term in the third equation to zero gives the condition of equilibrium

$$\tfrac{1}{2}K_1\sin 2\Phi_o = Y_1\cos\Phi_o - Z_1\sin\Phi_o \quad (13)$$

B. Natural Frequencies of the Coupled Oscillations in a Circular Orbit

The frequency equation of free oscillations for case 1 is given by Eq. (7); the formulation for cases 2 and 3 can be combined. By assuming harmonic solutions

$$\psi = A\sin\lambda M, \quad \Phi = B\sin\lambda M, \quad \delta\theta = C\cos\lambda M \quad (14)$$

for Eq. (10) of case 2 and

$$\psi = A\sin\lambda M, \quad \theta = B\sin\lambda M, \quad \delta\Phi = C\cos\lambda M \quad (15)$$

for Eq. (12) of case 3, a single frequency determinant is obtained

$$|A(\lambda)| = \begin{vmatrix} a_{11}-\lambda^2 & a_{12} & a_{13}\lambda \\ a_{21} & a_{22}-\lambda^2 & a_{23}\lambda \\ a_{31}\lambda & a_{32}\lambda & a_{33}-\lambda^2 \end{vmatrix} = 0 \quad (16)$$

The oscillatory motion is stable if the cubic algebraic equation obtained from the frequency determinant

$$(\lambda^2)^3 + \beta(\lambda^2)^2 + \gamma(\lambda^2) + \delta = 0 \quad (17)$$

yields three real positive roots for λ^2. Hence, the conditions for stability of the linearized equations are

$$-4/27 < \bar{c}^2/\bar{b}^3 < 0, \quad (\bar{b} = \gamma - \beta^2/3, \quad \bar{c} = \delta - \beta\gamma/3 + 2\beta^3/27)$$
$$-\beta = a_{11} + a_{22} + a_{33} + a_{13}a_{31} + a_{23}a_{32} > 0 \quad (18)$$
$$\gamma = a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11} - a_{12}a_{23}a_{31} - a_{13}a_{32}a_{21} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} > 0 - \delta = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} > 0$$

C. Harmonic Response of the Linearized Systems

Considering the disturbing functions, simple harmonic functions, we let

$$u_x = u_1\sin sM, \quad u_y = u_2\sin sM, \quad u_z = u_3\sin sM \quad (19)$$

where the u_i and s are constants. The harmonic response of the linearized systems can be obtained immediately in the following:

1) Case 1 equilibrium state

From Eq. (6) the response of the case 1 equilibrium is

$$\psi = \psi(0)\cos(3K_3)^{1/2}M + [\psi'(0)/(3K_3)^{1/2}]\sin(3K_3)^{1/2}M + u_3\{\sin sM - [s/(3K_3)^{1/2}]\sin(3K_3)^{1/2}M\}/(3K_3 - s^2) \quad (20)$$

$$\Phi = \Phi_c + \Phi_p$$

$$\Phi_c = \sum_{i=1}^{2}(-1)^i\{[(\lambda_i{}^2 - d + bc)\Phi(0) + b\theta'(0)\cos\lambda_i M] + (1/\lambda_i)[(\lambda_1{}^2 - d)\Phi'(0) - bd\theta(0)]\sin\lambda_i M\}/(\lambda_2{}^2 - \lambda_1{}^2) \quad (21)$$

$$\Phi_p = [(d-s^2)u_1\sin sM - bsu_2\cos sM]/(\lambda_1{}^2 - s^2)(\lambda_2{}^2 - s^2) - \sum_{i=1}^{2}(-1)^i s[bu_2\cos\lambda_i M + (\lambda_i{}^2 - d)(u_1/\lambda_i)\sin\lambda_i M]/(\lambda_i{}^2 - s^2)(\lambda_2{}^2 - \lambda_1{}^2)$$

$$\theta = \theta_c + \theta_p$$

$$\theta_c = \sum_{i=1}^{2}(-1)^i\{[(\lambda_i{}^2 - a + bc)\theta(0) + c\Phi'(0)]\cos\lambda_i M + (1/\lambda_i)[(\lambda_i{}^2 - a)\theta'(0) - ac\Phi(0)]\sin\lambda_i M\}/(\lambda_2{}^2 - \lambda_1{}^2) \quad (22)$$

$$\theta_p = [(a-s^2)u_1\sin sM - scu_2\cos sM]/(\lambda_1{}^2 - s^2)(\lambda_2{}^2 - s^2) - \sum_{i=1}^{2}(-1)^i s[cu_1\cos\lambda_i M + (1/\lambda_i)(\lambda_i{}^2 - a)u_2\sin\lambda_i M]/(\lambda_i{}^2 - s^2)(\lambda_2{}^2 - s^2)$$

2) Case 2 and case 3 equilibrium states

With the aid of Table 1, Eqs. (10) and (12) can be represented by a single system of equations

$$\psi'' + a_{11}\psi + a_{12}\xi - a_{13}\eta' = R\sin sM$$
$$a_{12}\psi + \xi'' + a_{22}\xi - a_{23}\eta' = P\sin sM \quad (23)$$
$$a_{31}\psi' + a_{32}\xi' + \eta'' + a_{33}\eta = Q\sin sM$$

If the determinant $D = |A(s)|$, as defined by Eq. (16) is not zero, the harmonic response of these two cases can be readily obtained

Table 1 Tabulated definitions

Symbols	ξ	η	P	Q	R
Case 2	Φ	$\delta\theta$	$u_1 + u_3\tan\theta_o$	u_2	$u_3\sec\theta_o$
Case 3	θ	$\delta\Phi$	$u_2\cos\Phi_o - u_3\sin\Phi_o$	u_1	$u_2\sin\Phi_o + u_3\cos\Phi_o$

using Laplace transform. The solution of Eq. (23) in matrix form is

$$\begin{vmatrix} \psi \\ \xi \\ \eta \end{vmatrix} = \sum_{k=1}^{3} \{[(A)_k - s(B)_k] \cos \lambda_k M + [(\bar{A})_k - s(\bar{B})_k] \sin \lambda_k M\} - s(C)_1 \cos sM - (C)_2 \sin sM \tag{24}$$

where the constant column matrices are defined in Appendix C.

D. First-Order Eccentricity Forced Motion

The term $2e \sin M$ appearing in the first equation of each system of equations given by Eqs. (6), (10), and (12) corresponds to a forcing function due to orbital eccentricity. As mentioned earlier, the eccentricity is assumed to have the same order of magnitude as the angular variables. Response to this forcing function can be obtained directly from the formulas for the harmonic response. For case 1 equilibrium state we obtain $\Phi = \Phi_c$ and $\theta = \theta_c$ given by Eqs. (21) and (22), respectively, and ψ from Eq. (20) by putting $s = 1$ if $3K_3 \neq 1$. It is noted that Φ and θ are not affected by the forcing function through linear coupling with ψ. For cases 2 and 3 equilibrium states we obtain the eccentricity forced motion by setting $s = 1$, $P = Q = 0$, $R = 2e$ in Eq. (24) if $\lambda_k \neq 1$ $(k = 1, 2, 3)$.

E. Second-Order Eccentricity Forced Motion

To determine the effect of the orbital eccentricity on the motions of Φ and θ for the case 1 equilibrium state, the second-order terms in Eq. (4) must be retained in the derivation of the variational equations from that state. We now attempt series solutions for the variables. Let us assume the forced motion of ψ in the form

$$\psi = a_1 e \sin M + a_2 e^2 \sin 2M + \cdots \tag{25}$$

where the constant coefficients a_1 and a_2 will be determined later. The variables Φ and θ are expressed in power series of the eccentricity with undetermined functions as coefficients

$$\begin{aligned} \Phi &= e\Phi_1 + e^2\Phi_2 + \cdots \\ \theta &= e\theta_1 + e^2\theta_2 + \cdots \end{aligned} \tag{26}$$

The differential equations for the undetermined functions are obtained by direct substitution of Eqs. (5), (25), and (26) into the second and third equations of Eq. (4) and collection of terms of equal power in e. This leads to

$$\begin{aligned} \Phi_1'' + b\theta_1' + a\Phi_1 &= 0 \\ \theta_1'' + c\Phi_1' + d\theta_1 &= 0 \end{aligned} \tag{27}$$

and

$$\begin{aligned} \Phi_2'' + b\theta_2' + a\Phi_2 &= (1 - K_1)(2 + a_1)\theta_1 \cos M - \\ &\quad (K_1 + K_3)a_1\theta_1 \sin M - (2K_1 + Z_1)(2 + a_1)\Phi_1 \cos M \\ \theta_2'' + c\Phi_2' + d\theta_2 &= 3(K_2 + K_3)a_1\Phi_1 \sin M - \\ &\quad (1 + K_2)(2 + a_1)\Phi_1 \cos M + [K_2(13 + 2a_1) - (2 + a_1)Z_2]\theta_1 \cos M \end{aligned} \tag{28}$$

We note that Eq. (27) represents the coupled free oscillations, and the solutions are $e\Phi_1 = \Phi_c$ and $e\theta_1 = \theta_c$ given by Eqs. (21) and (22), respectively.

Since the homogeneous equations of Eq. (28) are identical with Eq. (27), the homogeneous solutions of Φ_2 and θ_2 may be absorbed in the solutions of Φ_1 and θ_1. We may write the particular solutions of Eq. (28) in the forms

$$\begin{aligned} \Phi_2 &= \sum_{k=1}^{4} (b_k \cos s_k M + c_k \sin s_k M) \\ \theta_2 &= \sum_{k=1}^{4} (d_k \cos s_k M + e_k \sin s_k M) \\ s_{1,2} &= \lambda_1 \pm 1, \qquad s_{3,4} = \lambda_2 \pm 1 \end{aligned} \tag{29}$$

where constants b, c, d, and e can be obtained by direct substitution of Eq. (29) into Eq. (28) and the results are omitted here.

The algebraic equation for the determination of a_1 and a_2 in Eq. (25) are obtained by using Galerkin's method which may be written in the form

$$\int_0^{2\pi} L(\psi) \sin jMdM = 0, \qquad j = 1, 2 \tag{30}$$

where $L(\psi) = 0$ represents the first equation of Eq. (4). With the aid of the following notations

$$\begin{aligned} I_j(f(M)) &= \int_0^{2\pi} f(M) \sin jMdM \\ E_{1j} = I_j(\Phi_1\theta_1), &\qquad E_{2j} = I_j(\Phi_1\theta_1') \\ E_{3j} = I_j(\theta_1\theta_1'), &\qquad E_{4j} = I_j(\Phi_1'\theta_1') \end{aligned} \tag{31}$$

Eq. (30) results in

$$\begin{aligned} a_1 = \{2 + e[(4K_2 + 4K_3 - Z_2)E_{11} + (K_2 + K_3 - Z_2)E_{21} - \\ (1 + K_3)E_{31} - (1 - K_3)E_{41}]\}/(3K_3 - 1) \end{aligned} \tag{32}$$

$$\begin{aligned} a_2 = [5 - 4.5a_1^2 + (4K_2 + 4K_3 - Z_2)E_{12} - (K_2 + K_3 - Z_2)E_{22} + \\ (1 + K_3)E_{32} + (1 - K_3)E_{42}]/(3K_3 - 1) \end{aligned} \tag{33}$$

if $K_3 \neq \frac{1}{3}$.

The second-order eccentricity forced motions for cases 2 and 3 can be formulated by using the same procedure as that used for case 1. The second-order solution is quite lengthy and is omitted here.

E. Dynamical Coupling with an Elastic Appendage

Let us consider that the primary body consists of a rigid main body and an elastic appendage attached to the main body at point P. Vector R denotes the position of P from the center of mass of the primary body and ξ, η, ζ are body-fixed coordinates of the appendage with the origin at P. To simplify the analysis, the following assumptions are made: 1) the mass of the appendage is much smaller than the mass of the main body, such that the elastic deformation of the appendage has negligible effect on the center of mass and the gravity-gradient moment of the system; 2) the transverse vibration of the appendage is in the ξ–η plane and it is expressed by

$$u(\xi, t) = U(\xi)\tau(t) \tag{34}$$

where the modal function $U(\xi)$ is an assumed function and $\tau(t)$ is a generalized coordinate to be determined; and 3) the orbital coordinate $x_o y_o z_o$ may be considered as an inertial coordinate when we deal with the rotational motion of a satellite. Consequently, the transverse vibration of the appendage is not coupled with the orbital motion.

The dynamical coupling of an elastic appendage with the main body may be treated by making the disturbing torques in Eq. (4)

$$T_x = -\dot{h}_{ex}, \qquad T_y = -\dot{h}_{ey}, \qquad T_z = -\dot{h}_{ez} \tag{35}$$

where $\dot{h}_e$ is the time derivative of the angular momentum vector due to elastic motion, and the expression for $\dot{h}_e$ is given by Eq. (B-10). As presented in Appendix B, the equation for the generalized coordinate, τ, obtained by using Lagrange's equation is

$$m\ddot{\tau} + k\tau = -(c_1\dot{\omega}_x + c_2\dot{\omega}_y + c_3\dot{\omega}_z) \tag{36}$$

Thus, Eqs. (4) and (36) present a system of four equations for the variables, ψ, Φ, θ, and τ. We take case 1 equilibrium for an illustration. The linearized system of equations are

$$\begin{aligned} \psi'' + 3K_3 &= -c_3\tau''/I_z \\ \Phi'' + b\theta' + a\Phi &= -(c_1\tau'' - c_2\tau')/I_x \\ \theta'' + c\Phi' + d\theta &= -(c_2\tau'' + c_1\tau')/I_y \\ \bar{m}\tau'' + (k/n_o^2)\tau &= -c_1\Phi'' + c_2(\theta'' + \Phi') + c_3\psi'' \end{aligned} \tag{37}$$

Example 1

Let us consider that a uniform, slender bar is attached to the main body at its midpoint and let the $\xi\eta\zeta$ coordinates coincide with the xyz coordinates. We take $e_1 = j$, $e_2 = k$, $e_3 = i$, and $R = L_1 i + L_3 k$. From Eqs. (B-3–B-5) we obtain $c_1 = a_2$,

$c_2 = a_1 L_1$, and $c_3 = 0$. For a symmetric bending mode, $a_2 = 0$, Eq. (37) yields the frequency determinant

$$\begin{vmatrix} a-\lambda^2 & -b\lambda & c_2\lambda/I_x \\ c\lambda & d-\lambda^2 & -c_2\lambda^2/I_y \\ -c_2\lambda & -c_2\lambda^2 & k/n_o{}^2-\bar{m}\lambda^2 \end{vmatrix} = 0 \qquad (38)$$

For a skew-symmetric bending mode, $a_1 = 0$, the system of equations gives the frequency determinant

$$\begin{vmatrix} a-\lambda^2 & -b\lambda & -a_2\lambda^2/I_x \\ c\lambda & d-\lambda^2 & a_2\lambda/I_y \\ -a_2\lambda^2 & 0 & k/n_o{}^2-\bar{m}\lambda^2 \end{vmatrix} = 0 \qquad (39)$$

Example 2

If the bar is parallel to the x-axis, we have $e_1 = i$, $e_2 = k$, $e_3 = -j$, $c_1 = 0$, $c_2 = -a_1 L_1 - a_2$, $c_3 = 0$. The frequency determinant for this example has an identical form to the symmetric mode case of example 1.

Example 3

A symmetric cross is attached to the main body at $R = L_1 i + L_3 k$ and makes 45° with the x-axis. We may use Eq. (38), in which c_2 is replaced by $2c_2$ for the frequency equation of a symmetric bending mode and use Eq. (39), in which a_2 is replaced by $(2a_2)^{1/2}$, for a skew-symmetric bending mode.

III. Conclusions

Equations of the attitude motions of gyrostat satellite have been derived by method of perturbation. The linearized systems of equations for cases 2 and 3 equilibria can be combined into a single system. Frequency determinant, conditions of stable motions and harmonic response for the linearized system are given. For a satellite in an elliptical orbit the first-order and second-order of the eccentricity forced motions have been obtained. Transverse vibration of a flexible appendage is treated by assumed-mode method and its interaction with the attitude motions is formulated by using D'Alembert's principle. It can be seen that additional vibration modes and more flexible members can be added to the system with ease.

Appendix A: The Constants a_{ij}

The constants a_{ij} used in Eq. (10) are

$$a_{11} = 3K_3, \quad a_{12} = -3(K_2+K_3)\sin\theta_o + Z_2\tan\theta_o + X_2 - X_3$$

$$a_{13} = (1+K_3)\tan\theta_o - X_3\sec\theta_o, \quad \alpha_{21} = 3(K_1+K_3)\sin\theta_o$$

$$a_{22} = -[3(K_2+K_3)\sin\theta_o - X_2 + X_3 - Z_2\tan\theta_o]\sin\theta_o + K_1(1-4\sin^2\theta_o) + Z_1\cos\theta_o$$

$$a_{23} = \sec\theta_o - X_3\tan\theta_o + K_3\sin\theta_o\tan\theta_o - K_1\cos\theta_o - Z_1$$

$$a_{31} = K_2\sin 2\theta_o$$

$$a_{32} = (1+K_2)\cos\theta_o - Z_2$$

$$a_{33} = -4K_2\cos 2\theta_o + Z_2\cos\theta_o - X_2\sin\theta$$

The constants a_{ij} in Eq. (12) are defined as

$$a_{11} = 3(K_3\cos^2\Phi_o - K_2\sin^2\Phi_o)$$

$$a_{12} = -2(K_2+K_3) - Y_3\cos\Phi_o + Z_2\sin\Phi_o$$

$$a_{13} = \tfrac{1}{2}(K_3+K_2) + Y_3\cos\Phi_o - Z_2\sin\Phi_o$$

$$a_{21} = \tfrac{3}{2}(K_3+K_2)\sin 2\Phi_o$$

$$a_{22} = 4(K_3\sin^2\Phi_o - K_2\cos^2\Phi_o) + Z_2\cos\Phi_o + Y_3\sin\Phi_o$$

$$a_{23} = K_3\sin^2\Phi_o - K_2\cos^2\Phi_o - Z_2\cos\Phi_o - Y_3\sin\Phi_o$$

$$a_{31} = \tfrac{1}{2}K_1\sin 2\Phi_o, \qquad a_{32} = -(1+a_{33})$$

$$a_{33} = K_1\cos 2\Phi_o + Y_1\sin\Phi_o + Z_1\cos\Phi_o$$

Appendix B: Formulation for an Elastic Appendage

The strain energy for bending of a flexible appendage, as described in Section II-E is

$$V = \tfrac{1}{2}k\tau^2$$

where the generalized spring constant k is defined by

$$k = \frac{1}{2}\int EI\left(\frac{\partial^2 U}{\partial \xi^2}\right)^2 d\xi \qquad \text{(B-1)}$$

and EI is the bending stiffness of the appendage. The kinetic energy for the system is

$$T = \tfrac{1}{2}\textstyle\int_D (v_r+v_e)\cdot(v_r+v_e)\,dm \qquad \text{(B-2)}$$

where D represents the domain of the main body and its appendage, v_r and v_e are velocities due to rigid-body rotation and elastic deformation, respectively. We write the unit vectors of the $\xi\eta\zeta$ coordinates in terms of the unit vectors of the xyz coordinates

$$\begin{aligned} e_1 &= \alpha_1 i + \alpha_2 j + \alpha_3 k \\ e_2 &= \beta_1 i + \beta_2 j + \beta_3 k \\ e_3 &= \gamma_1 i + \gamma_2 j + \gamma_3 k \end{aligned} \qquad \text{(B-3)}$$

and denote

$$\begin{aligned} a_1 &= \textstyle\int_A U(\xi)\rho(\xi)\,d\xi \\ a_2 &= \textstyle\int_A U(\xi)\rho(\xi)\xi\,d\xi \\ \bar{m} &= \textstyle\int_A U^2(\xi)\rho(\xi)\,d\xi \end{aligned} \qquad \text{(B-4)}$$

where $\rho(\xi)$ is the mass density function of the appendage and A represents the span of the appendage in ξ direction. Let us define

$$\begin{aligned} c_1 &= a_1 R\times e_2\cdot i + a_2\gamma_1 = a_1 R\cdot(\beta_3 j - \beta_2 k) + a_2\gamma_1 \\ c_2 &= a_1 R\times e_2\cdot j + a_2\gamma_2 = a_1 R\cdot(\beta_1 k - \beta_3 i) + a_2\gamma_2 \\ c_3 &= a_1 R\times e_2\cdot k + a_2\gamma_3 = a_1 R\cdot(\beta_2 i - \beta_1 j) + \alpha_2\gamma_3 \end{aligned} \qquad \text{(B-5)}$$

With the aid of Eqs. (B-2–B-5), the kinetic energy of the appendage is

$$T = T_r + \tfrac{1}{2}\bar{m}\dot{\tau}^2 + \dot{\tau}(c_1\omega_x + c_2\omega_y + c_3\omega_z) \qquad \text{(B-6)}$$

where T_r is the kinetic energy due to rigid-body rotation. It follows from Lagrange's equation that the equation for τ is

$$\bar{m}\ddot{\tau} + k\tau = -(c_1\dot{\omega}_x + c_2\dot{\omega}_y + c_3\dot{\omega}_z) \qquad \text{(B-7)}$$

Next, we formulate the angular momentum vector, h, and its time derivative. We have

$$\begin{aligned} h &= \textstyle\int_D r\times(v_r+v_e)\,dm = h_r + h_e \\ h_e &= \dot{\tau}(c_1 i + c_2 j + c_3 k) \end{aligned} \qquad \text{(B-8)}$$

where h_r and h_e are angular momentum vectors due to rigid-body rotation and elastic deformation, respectively. Differentiation of h_e with respect to time results in

$$\dot{h}_e = \ddot{\tau}(c_1 i + c_2 j + c_3 k) + \omega\times h_e = \dot{h}_{ex} i + \dot{h}_{ey} j + \dot{h}_{ez} k \qquad \text{(B-9)}$$

where

$$\dot{h}_{ex} = c_1\ddot{\tau} + \dot{\tau}[(\dot{\psi}+\dot{f})\cos\theta(c_3\sin\Phi - c_2\cos\Phi) + \dot{\theta}(c_3\cos\Phi + c_2\sin\Phi)]$$

$$\dot{h}_{ey} = c_2\ddot{\tau} + \dot{\tau}[(\dot{\psi}+\dot{f})(c_1\cos\theta\cos\Phi + c_3\sin\theta) - c_1\dot{\theta}\sin\Phi - c_3\dot{\Phi}] \qquad \text{(B-10)}$$

$$\dot{h}_{ez} = c_3\ddot{\tau} - \dot{\tau}[(\dot{\psi}+\dot{f})(c_1\cos\theta\sin\Phi + c_2\sin\theta) + c_1\dot{\theta}\cos\Phi - c_2\dot{\Phi}]$$

Appendix C: Notations of Equation (24)

The constant column matrices used in Eq. (24) are defined as follows:

$$(A)_k = [(\alpha_1) - (\alpha_3)\lambda_k{}^2 + (\alpha_5)\lambda_k{}^4]/D_k$$

$$(\bar{A}) = [(\alpha_0) - (\alpha_2)\lambda_k{}^2 + (\alpha_4)\lambda_k{}^4]/D_k\lambda_k(\lambda_k{}^2 - s^2)$$

$$(B)_k = [(\beta_1) - (\beta_3)\lambda_k{}^2]/D_k$$

$$(\bar{B})_k = [(\beta_0) - (\beta_2)\lambda_k{}^2 + (\beta_4)\lambda_k{}^4]/D_k\lambda_k(\lambda_k{}^2 - s^2)$$

$$(C)_1 = [(\beta_1) - (\beta_3)s^2]/D$$

$$(C)_2 = [(\beta_0) - (\beta_2)s^2 + (\beta_4)s^2]/D$$

$$D_k = (\lambda_k{}^2 - \lambda_i{}^2)(\lambda_k{}^2 - \lambda_j{}^2)(i, j, k = 1, 2, 3, i \neq j, i \neq k, j \neq k)$$

$$D = (s^2 - \lambda_1{}^2)(s^2 - \lambda_2{}^2)(s^2 - \lambda_3{}^2)$$

The constants α and β in the above column matrices are respectively,

1) For the variable ψ

$$\alpha_0 = a_{22}a_{33}\psi'(0) - a_{12}a_{33}\xi'(0) - a_{33}(a_{13}a_{22} - a_{12}a_{23})\eta(0)$$

$$\alpha_1 = a_{22}a_{33}\psi'(0) + (a_{13}a_{22} - a_{12}a_{23})\eta'(0) + [a_{32}(a_{13}a_{22} - a_{12}a_{23}) - a_{12}a_{33}]\xi(0) + a_{31}(a_{13}a_{22} - a_{12}a_{23})\psi(0) - a_{13}a_{22}a_{33}\eta(0)$$

$$\alpha_2 = (a_{22}a_{33} + a_{23}a_{32})\psi'(0) - (a_{12} + a_{13}a_{32})\xi'(0) + a_{13}a_{22}(1 - a_{33})\eta(0)$$

$$\alpha_3 = (a_{13}a_{31} + a_{22} + a_{33} + a_{23}a_{32})\psi(0) + a_{13}\eta'(0) - a_{12}\xi(0)$$

$$\alpha_4 = \psi'(0) \qquad \alpha_5 = \psi(0)$$

$$\beta_0 = a_{22}a_{33}R - a_{12}a_{33}P \qquad \beta_1 = (a_{13}a_{22} - a_{12}a_{23})Q$$

$$\beta_2 = (a_{22} + a_{33} + a_{23}a_{32})R - (a_{13}a_{32} + a_{12})P$$

$$\beta_3 = a_{13}Q \qquad \beta_4 = R$$

2) For the variable ξ

$$\alpha_0 = a_{11}a_{33}\xi'(0) - a_{21}a_{33}\psi'(0) - (a_{11}a_{23} - a_{13}a_{21})a_{33}\eta(0)$$

$$\alpha_1 = [a_{11}a_{33} + a_{32}(a_{11}a_{23} - a_{13}a_{21})]\xi(0) + [a_{31}(a_{11}a_{23} - a_{13}a_{21}) - a_{21}a_{33}]\psi(0) + (a_{11}a_{33} - a_{13}a_{21})\eta'(0)$$

$$\alpha_2 = (a_{11} + a_{33} + a_{13}a_{31})\xi'(0) - (a_{23}a_{31} + a_{21})\psi'(0) - a_{23}a_{33}\eta(0)$$

$$\alpha_3 = a_{23}\eta'(0) + (a_{11} + a_{33} + a_{13}a_{31} + a_{23}a_{32})\xi(0) - a_{21}\psi(0)$$

$$\alpha_4 = \xi'(0) \qquad \alpha_5 = \xi(0)$$

$$\beta_0 = a_{11}a_{33}P - a_{21}a_{33}R \qquad \beta_1 = (a_{11}a_{23} - a_{13}a_{21})Q$$

$$\beta_2 = (a_{11} + a_{33} + a_{13}a_{31})P - (a_{23}a_{31} + a_{21})R$$

$$\beta_3 = a_{23}Q \qquad \beta_4 = P$$

3) For the variable η

$$\alpha_0 = (a_{11}a_{22} - a_{12}a_{21})[\eta'(0) + a_{31}\psi(0) + a_{32}\xi(0)]$$

$$\alpha_1 = (a_{21}a_{32} - a_{22}a_{31})\psi'(0) + (a_{12}a_{31} - a_{11}a_{32})\xi'(0) + [a_{11}a_{22} - a_{12}a_{21} - a_{13}(a_{21}a_{32} - a_{22}a_{31}) - a_{23}(a_{12}a_{31} - a_{11}a_{32})]\eta(0)$$

$$\alpha_2 = (a_{11} + a_{22})\eta'(0) + (a_{11}a_{31} + a_{21}a_{32})\psi(0) + (a_{12}a_{31} + a_{22}a_{32})\xi(0)$$

$$\alpha_3 = (a_{11} + a_{22} + a_{13}a_{31} + a_{23}a_{32})\eta(0) - a_{31}\psi'(0) - a_{32}\xi'(0)$$

$$\alpha_4 = \eta'(0) \qquad \alpha_5 = \eta(0)$$

$$\beta_0 = (a_{11}a_{22} - a_{12}a_{21})Q$$

$$\beta_1 = (a_{12}a_{32} - a_{11}a_{32})P + (a_{21}a_{32} - a_{22}a_{31})R$$

$$\beta_2 = (a_{11} + a_{22})Q \qquad \beta_3 = -a_{31}R - a_{32}P \qquad \beta_4 = Q$$

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